

TYPICAL POINTS AND FAMILIES OF EXPANDING INTERVAL MAPPINGS

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ABSTRACT. We study parametrised families of piecewise expanding interval mappings $T_a: [0, 1] \rightarrow [0, 1]$ with absolutely continuous invariant measures μ_a and give sufficient conditions for a point $X(a)$ to be typical with respect to (T_a, μ_a) for almost all parameters a . This is similar to a result by D. Schnellmann, but with different assumptions.

1. INTRODUCTION

Assume that I is an interval of parameters, and that for any $a \in I$ we have a mapping $T_a: [-1, 1] \rightarrow [-1, 1]$ that is piecewise expanding in a smooth and uniform way, and that T_a depends on a in a smooth way. It is a well known and classical result that such mappings have invariant measures that are absolutely continuous with respect to Lebesgue measure.

In [2], Schnellmann studied a class of such mappings together with a point $X(a)$ in the domain of T_a . Under some conditions on the family of mappings and on the function $a \mapsto X(a)$, he proved that for almost all $a \in I$, the point $X(a)$ is typical with respect to an invariant measure of T_a that is absolutely continuous with respect to Lebesgue measure. We say that a point x is typical with respect to (T, μ) if the sequence of measures $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_a^k(x)}$ converges weakly to μ as $n \rightarrow \infty$.

In this paper we consider similar kind of mappings T_a as Schnellmann and prove a corresponding result, that $X(a)$ is typical for almost all parameters a , but with assumptions that are different than those used by Schnellmann. The proof of this result is based on the proof by Schnellmann, but contains a new ingredient, that makes it possible to remove one of the more restrictive assumptions used by Schnellmann. Unfortunately, the proof also needs some new assumptions on the mapping, so the result of this paper is not a generalisation of Schnellmann's result, but it extends the result to some families of mappings that were not covered by Schnellmann.

Almost sure typicality for families of piecewise expanding interval mappings has also been proved by Schnellmann in [3, Theorem 3.5], but with different assumptions and with a method different from that used in [2]. Because of assumption (III) used by Theorem 3.5 in [3], it is somewhat unclear in what generality the result holds. (See also Remark 4.2 of [3].) Schnellmann proves that the assumption (III) is satisfied for families of tent mappings, generalised β -transformations and Markov mappings [3, Section 3],

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but for other classes, the validity of assumption (III) is unclear. In this paper we will prove results for families that are not contained in Schnellmann's papers [2, 3].

2. STATEMENT OF THE RESULT

We start by stating more precisely what kind of mappings we will work with. As mentioned before, I is an interval of parameters, and for any $a \in I$ we have a mapping $T_a: [-1, 1] \rightarrow [-1, 1]$ that satisfies the following.

- There are smooth functions b_0, \dots, b_p with

$$-1 = b_0(a) < b_1(a) < \dots < b_p(a) = 1$$

for each $a \in I$, such that the restriction of T_a to $(b_i(a), b_{i+1}(a))$ can be extended to a smooth and monotone function on some open neighbourhood of $[b_i(a), b_{i+1}(a)]$.

- There are numbers $1 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \leq |T'_a(x)| \leq \Lambda$$

holds for all $a \in I$ and all $x \in [0, 1] \setminus \{b_0(a), \dots, b_p(a)\}$. There is a number L such that T'_a is Lipschitz continuous with constant L on each $[b_i(a), b_{i+1}(a)]$.

- For $x \in [0, 1]$, the mappings $a \mapsto T_a(x)$ and $a \mapsto T'_a(x)$ are piecewise C^1 .

For each a there is a T_a -invariant probability measure that is absolutely continuous with respect to Lebesgue measure, and there are at most finitely many such measures. In this paper we will work with mappings for which there is a unique absolutely continuous invariant probability measure μ_a . Let $K(a) = \text{supp } \mu_a$. By Wong [8] or Kowalski [4], $K(a)$ consists of finitely many intervals. Schnellmann assumed that the endpoints of these intervals depend in a smooth way on a . We will assume that $K(a) = [-1, 1]$ for all a . (More precisely, we assume that Assumption 5 below holds, which implies $K(a) = [-1, 1]$.) This is not too restrictive, since this can be achieved by restricting the mapping to $K(a)$ and a change of variable. Smoothness of the family will then be preserved if $K(a)$ changes in a smooth way with the parameter a .

For a piecewise continuous mapping $T: [-1, 1] \rightarrow [-1, 1]$ we let $\mathcal{P}(T)$ denote the partition of $[-1, 1]$ into the maximal open intervals on which T is continuous. We also write $\mathcal{P}_j(a) = \mathcal{P}(T_a^j)$ and denote by $-1 = b_{0,j}(a) < b_{1,j}(a) < \dots < b_{p_j,j}(a) = 1$ the points such that $\mathcal{P}_j(a) = \{(b_{i,n}(a), b_{i+1,n}(a))\}$.

We shall study the orbit of a point $X(a)$, and we write

$$\xi_j(a) = T_a^j(X(a)).$$

We assume throughout that X is a C^1 function. Hence, the function ξ_j is piecewise smooth. We denote by \mathcal{Q}_j the set of maximal open intervals on which ξ_j is smooth, that is the maximal open intervals such that $\xi_i(a) \notin \{b_0(a), \dots, b_p(a)\}$ holds for $0 \leq i < j$.

We state below four assumptions on the family of mappings and the function X . These conditions are the same as those used by Schnellmann in [2].

Assumption 1. There is a constant C_0 such that for any $j \geq 1$ and any $\omega \in \mathcal{Q}_j$ we have

$$C_0^{-1} \leq \left| \frac{\xi'_j(a)}{(T_a^j)'(X(a))} \right| \leq C_0$$

for all $a \in \omega$.

It may be difficult to check if Assumption 1 hold. Schnellmann proved that the following assumption implies Assumption 1. (See Lemma 2.1 in [2].)

Assumption 2. X is C^1 , and there is a j_0 such that

$$\inf_a |\xi'_{j_0}(a)| \geq \frac{\sup_{a,x} |\partial_a T_a(x)|}{\lambda - 1} + 2L.$$

Let ϕ_a denote the density of the absolutely continuous measure μ_a . Our next assumption concerns this density.

Assumption 3. There is a constant C_1 such that for all $a \in I$ we have

$$C_1^{-1} \leq \phi_a \leq C_1, \quad \mu_a\text{-a.e.}$$

As is mentioned by Schnellmann in [2], the next assumption is more restrictive than the above assumptions. It is the purpose of the paper to extend Schnellmann's result to families of mappings that do not necessarily satisfy this assumption.

Assumption 4. There is a constant C_2 such that for all $a_1, a_2 \in I$, $a_1 \leq a_2$, and $j \geq 1$, there is a mapping

$$\mathcal{U}_{a_1, a_2, j}: \mathcal{P}_j(a_1) \rightarrow \mathcal{P}_j(a_2),$$

such that $\omega \in \mathcal{P}_j(a_1)$ and $\mathcal{U}_{a_1, a_2, j}(\omega)$ have the same symbolic dynamics, their images lie close in the sense that

$$d(T_{a_1}^j(\omega), T_{a_2}^j(\mathcal{U}_{a_1, a_2, j}(\omega))) \leq C_2 |a_1 - a_2|,$$

and

$$|T_{a_1}^j(\omega)| \leq C_2 |T_{a_2}^j(\mathcal{U}_{a_1, a_2, j}(\omega))|.$$

Schnellmann proved that if the Assumptions 1, 3 and 4 are satisfied, then the point $X(a)$ is typical for (T_a, μ_a) for Lebesgue almost every $a \in I$. Assumption 4 is rather restrictive, and it would be desirable to remove this assumption. We shall do so, but in doing so, we will need to introduce the following two conditions instead.

If for any $\omega \in \mathcal{P}(a)$ there is an $N = N(a)$ such that

$$[-1, 1] \setminus \bigcup_{n=0}^N T_a^n(\omega)$$

is a finite set, then the mapping T is called weakly covering by Liverani in [5]. According to Lemma 4.2 of that paper, weakly covering implies that the mapping has a unique absolutely continuous invariant measure, with density that is bounded and bounded away from zero, and it is possible to give an explicit lower bound on the density. We shall need such lower bounds that are also stable under certain perturbations of the mapping. In order to achieve this we will need to do as follows.

For $\omega \in \mathcal{P}(a)$, let $\tilde{T}_a(\omega) = T_a(\omega)$. Suppose that $\tilde{T}_a^k(\omega)$ is defined. Then we define

$$\tilde{T}_a^{k+1}(\omega) = \bigcup_{\substack{\omega' \in \mathcal{P}(a), \\ \omega' \subset \tilde{T}_a^k(\omega)}} T_a(\omega').$$

Note that $\tilde{T}_a^k(\omega) \subset T_a^k(\omega)$.

Our next assumption will be the following assumption that is stronger than weakly covering.

Assumption 5. For any $\omega \in \mathcal{P}(a)$ there is an $N = N(a)$ such that

$$[-1, 1] \setminus \bigcup_{n=0}^N \tilde{T}_a^n(\omega)$$

is a finite set.

Assumption 6. There is a number $\delta > 0$ and an integer $m \geq 1$ such that

$$(-\delta, \delta) \subset T_a^m(b_{i,m}(a), b_{i+1,m}(a))$$

for all i and $a \in I$, and

$$\delta > \frac{1}{\inf |(T_a^m)'| - 1}$$

for all $i \in I$.

We shall prove the following.

Theorem 1. *Suppose that the family T_a and the point $X \in C^1$ satisfies the Assumptions 2, 5 and 6. Then the point $X(a)$ is typical with respect to (T_a, μ_a) for Lebesgue almost every $a \in I$.*

3. EXAMPLE

Let $0 = b_0 < b_1 < b_2 < \dots$ be an increasing and unbounded sequence of real numbers. Suppose $T: [0, \infty) \rightarrow [0, 1]$ is smooth on each of the intervals (b_i, b_{i+1}) and $|T'(x)| \geq \lambda_0 \geq 1$. We then define a family of mappings $T_a: [0, 1] \rightarrow [0, 1]$ by $T_a(x) = T(ax)$.

Corollary 1. *Suppose that there exists a δ such that*

$$\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right) \subset T((b_i, b_{i+1}))$$

for all i . Let

$$a_0 = \lambda_0^{-1}(1 + \delta^{-1}).$$

If $I \subset [a_0, \infty)$ is an interval such that for some i we have

$$(b_i/a, b_{i+1}/a) \subset \left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right), \quad \forall a \in I,$$

and $T(b_i, b_{i+1}) = [0, 1]$, then $X(a)$ is typical for a.e. $a \in I$ provided either

i) $X'(a) \geq 0$ and T is piecewise increasing,

or

ii) X satisfies Assumption 2.

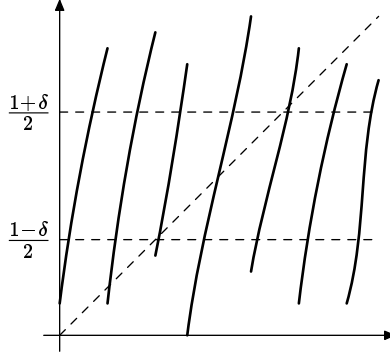


FIGURE 1. An example of a mapping T for which the assumptions in Corollary 1 are satisfied for $T_a(x) = T(ax)$, for all parameters in some interval $[1, a_1]$, $a_1 > 1$. Here we have taken $\delta = 2/5$. Assumption 6 is then that $\inf |T'_a| > 7/2$.

Schnellmann considered also families of the form $T_a(x) = T(ax)$, see Section 1.1 in [2], but used the assumption that $T((b_i, b_{i+1}))$ is always of the form $[0, c)$, and that T is piecewise expanding. Hence we can relax this assumption, but we have to add the other assumptions instead. Figure 1 shows an example of a mapping satisfying the assumptions of the corollary.

Proof of Corollary 1. We shall use Theorem 1. Note that Theorem 1 is for mappings on $[-1, 1]$ and that here we are working on the interval $[0, 1]$. This is just a matter of a change of variables.

In case $X'(a) \geq 0$ and T is piecewise increasing, Assumption 2 follows exactly as in Schnellmann's paper.

The assumptions on (b_i, b_{i+1}) and $(b_i/a, b_{i+1}/a)$ in the corollary implies that Assumption 5 holds. Assumption 6 is satisfied, since the definition of a_0 implies that $\delta > 1/(\inf |T'_a| - 1)$ for $a \in I$. Hence, the conclusion follows from Theorem 1. \square

4. OUTLINE OF THE PROOF

The proof goes along the same line as the proof of Schellmann. Let B be an interval and define

$$F_n(a) = \frac{1}{n} \sum_{j=1}^n \chi_B(\xi_j(a)).$$

It is sufficient to show that there exists a constant C such that

$$\limsup_{n \rightarrow \infty} F_n(a) \leq C|B|$$

holds for any interval B with rational endpoints. Using a lemma by Björklund and Schnellman [1] one shows that it is sufficient to show that

$$\int_{\tilde{I}} \chi_B(\xi_{j_1}(a)) \cdots \chi_B(\xi_{j_h}(a)) da \leq (C|B|)^h,$$

for sufficiently sparse sequences j_1, \dots, j_h , where h is an integer and $\tilde{I} \subset I$ is a small interval of parameters. These estimates are stated in Proposition 2

below, and are achieved by switching from an integral over the parameter space to an integral over the phase space for a fixed mapping.

This is where the main difference with Schnellmann's paper appears. Schnellmann switches the integral over the parameter space to an integral over the phase space for a mapping T_{a_J} in the family. In order to be able to do so it is important that the orbit structure of T_{a_J} is rich enough to contain the orbits of T_a for parameters a that are close to a_J . This is where Assumption 4 is necessary.

Here we will instead perturb the mapping T_{a_J} to a mapping that is close to T_{a_J} but does not belong to the family. In this way we can artificially make sure that a variant of Assumption 4 is satisfied, even if the assumption itself is not satisfied for the family. To prove that this is possible requires some new assumptions. In effect, we can remove Assumption 4, but we have to replace it with the assumptions used in Theorem 1.

In Section 5 below, we will state and prove the results that are necessary to get the desired properties of the above mentioned perturbation of the mapping T_{a_J} . In Section 6, we will prove Theorem 1.

5. SOME PREPARATIONS: ON NESTED SUBSHIFTS

Let $D = [-1, 1]$. We will consider piecewise expanding mappings on D .

Given a vector of numbers $b = (b_0, b_1, \dots, b_n)$ with $-1 = b_0 < b_1 < \dots < b_n = 1$ and a vector of functions $f = (f_0, f_1, \dots, f_n)$ with $f_k: [b_{k-1}, b_k] \rightarrow D$, we define a mapping $S: (b, f) \mapsto T$, where T is a mapping $T: D \rightarrow D$ such that $T(x) = f_k(x)$ for $x \in (b_{k-1}, b_k)$. We leave T undefined at the points b_k , and let $D_k = (b_{k-1}, b_k)$.

With the mapping T , we associate the shift space

$$\Sigma(T) = \{i \in \{1, 2, \dots, n\}^{\mathbb{N}} : \exists x \in D, T^k(x) \in D_{i_k}\}.$$

Suppose we have smooth functions b_0, b_1, \dots, b_n defined on $[0, \varepsilon]$, such that

$$-1 = b_0(t) < b_1(t) < \dots < b_n(t) = 1$$

and

$$|b'_k(t)| \leq \zeta$$

for all $t \in [0, \varepsilon]$, where ζ is a fixed number. We will use the notation $b_t = (b_0(t), b_1(t), \dots, b_n(t))$ and $D_k(t) = (b_{k-1}(t), b_k(t))$.

Let f_1, f_2, \dots, f_n be smooth mappings with $f_k: [0, P) \times D \rightarrow D$ such that for any $t \in [0, P)$ we have

$$\begin{aligned} \lambda \leq |\partial_x f_k(t, x)| &\leq \Lambda, & \text{for all } x \in D_k(t), \\ |\partial_t f_k(t, x)| &\leq \eta, & \text{for all } x \in D_k(t), \end{aligned}$$

where $1 < \lambda \leq \Lambda < \infty$ and η are fixed numbers. We also assume that if $f_k(t_0, b_k(t_0)) \in \{-1, 1\}$ for some t_0 , then $f_k(t, b_k(t)) = f_k(t_0, b_k(t_0))$ for all t , and similarly if $f_k(t_0, b_{k-1}(t_0)) \in \{-1, 1\}$.

Actually, we only assume that the functions f_k are defined for (t, x) such that $x \in D_k(t)$. We will consider t as our parameter and write $f_{k,t}$ for the function $x \mapsto f_k(t, x)$, and we define the vector $f_t = (f_{1,t}, f_{2,t}, \dots, f_{n,t})$.

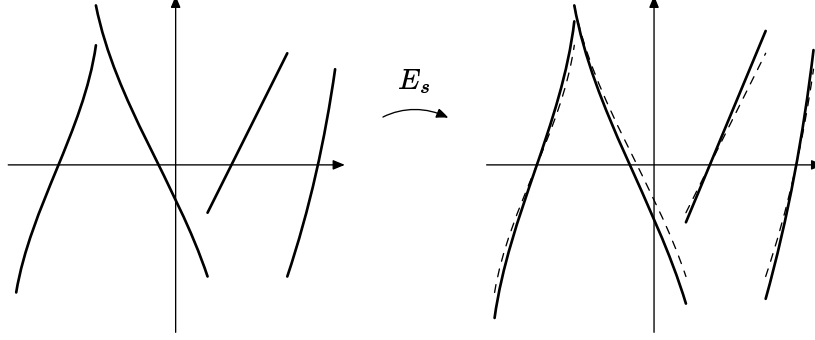


FIGURE 2. An illustration of the action of E_s with $s = 1.2$. The dashed lines show the original graph.

We now define mappings E_s . Given a mapping $f_k(t, \cdot): D_k(t) \rightarrow [-1, 1]$ and a number s close to one, we define

$$E_s(f_k(t, \cdot)) = \tilde{f}_k: D_k(t) \rightarrow [-1, 1],$$

where \tilde{f}_k is defined by

$$(1) \quad \tilde{f}_k(x) = \begin{cases} s f_k(t, x) & \text{if } \overline{f_k(t, D_k(t))} \subset (-1, 1), \\ f_k(t, x) & \text{if } f_k(t, D_k(t)) = (-1, 1), \\ \frac{s+1}{2} f_k(t, x) + \frac{s-1}{2} & \text{if } -1 \in \overline{f_k(t, D_k(t))} \subset [-1, 1), \\ \frac{s+1}{2} f_k(t, x) - \frac{s-1}{2} & \text{if } 1 \in \overline{f_k(t, D_k(t))} \subset (-1, 1]. \end{cases}$$

We shall only consider E_s for $s \geq 1$. Hence E_s maps $f_k(t, \cdot)$ into \tilde{f}_k , so that the graph of \tilde{f}_k is the graph of $f_k(t, \cdot)$, expanded in such a way that the image stays in $(-1, 1)$, see Figure 2.

When f_t is a vector $f_t = (f_{1,t}, f_{2,t}, \dots, f_{n,t})$ we define $E_s(f_t)$ by

$$E_s(f_t) = (E_s(f_{1,t}), E_s(f_{2,t}), \dots, E_s(f_{n,t})).$$

We now define the mappings $T_{t,s}: I \rightarrow I$ for any $t \in [0, P)$ and s close to one. Recall that $S: (b, f) \mapsto T$ was defined in the beginning of this section and define

$$T_{t,s} = S(b_t, E_s(f_t)).$$

We also define the symbolic spaces $\Sigma_{t,s}$ by

$$\Sigma_{t,s} = \Sigma(T_{t,s}).$$

We are going to prove the following.

Proposition 1. *Assume that $(-\delta, \delta) \subset f_k(t, D_k)$ for all k and t , where δ is a number such that*

$$\delta > \frac{1}{\lambda - 1}.$$

Then there are numbers α_0 and d , depending only on λ , Λ and ζ , such that

$$\left. \begin{array}{l} \alpha > \alpha_0 \\ 0 \leq t_0 < t_1 < d/\alpha \end{array} \right\} \Rightarrow \Sigma_{t_0, 1+\alpha t_0} \subset \Sigma_{t_1, 1+\alpha t_1}.$$

Moreover, we have

$$T_{t_0, 1+\alpha t_0}^j(\omega) \subset T_{t_1, 1+\alpha t_1}^j(\mathcal{U}_{t_0, t_1, j}(\omega))$$

for any $\omega \in \mathcal{P}_j(T_{t_0, 1+\alpha t_0})$, where $\mathcal{U}_{t_0, t_1, j}$ is as in Assumption 4.

The assumption that $(-\delta, \delta) \subset f_k(t, D_k(t))$ for any k is, as it is stated here, only here for convenience in the proof. One can make small adjustments in the definition of $T_{t,s}$ so that the conclusion of Proposition 1 holds with various variations of this assumption. It is not important that $0 \in f_k(t, D_k(t))$, but it seems that some kind of large image property is needed.

Proof. Fix t_0 and t_1 in $[0, \varepsilon) \subset [0, P)$ such that $t_0 < t_1$. Write $s(t) = 1 + \alpha t$.

It suffices to prove the following. For any $m \in \mathbb{N}$, $i \in \Sigma_{t_0, s(t_0)}$, and x_0, x_1, \dots, x_m such that

$$(2) \quad x_k = T_{t_0, s(t_0)}(x_{k-1}) = T_{t_0, s(t_0)}^k(x_0) \in D_{i_k}(t_0), \quad k = 1, 2, \dots, m,$$

there exist y_0, y_1, \dots, y_m such that

$$y_k = T_{t_1, s(t_1)}(y_{k-1}) = T_{t_1, s(t_1)}^k(y_0) \in D_{i_k}(t_1), \quad k = 1, 2, \dots, m.$$

Let $J_j(t) = T_{t, s(t)}(D_j(t)) = E_{s(t)}(f_{j,t}(D_j(t)))$. We could try to simply define y_k by

$$\begin{aligned} y_m &= x_m, \\ y_{k-1} &= (E_{s(t_1)}(f_{i_{k-1}, t_1}))^{-1}(y_k), \end{aligned}$$

but a possible obstruction is that y_k might not lie in $J_{i_{k-1}}(t_1)$, and in that case $(E_{s(t_1)}(f_{i_{k-1}, t_1}))^{-1}(y_k)$ is not defined. We will show that our assumptions imply that this obstruction is not present.

We let

$$\begin{aligned} y_m(t) &= x_m, \\ y_{k-1}(t) &= (E_{s(t)}(f_{i_{k-1}, t}))^{-1}(y_k(t)), \end{aligned}$$

and proceed by induction. Clearly, $y_0(t), y_1(t), \dots, y_m(t)$ are defined for $t = t_0$, and since the intervals $J_j(t)$ are open, it follows that $y_0(t), y_1(t), \dots, y_m(t)$ are defined in some neighbourhood around $t = t_0$. We want to estimate the size of that neighbourhood, and for future reference, we denote it by L .

Clearly, $y_m(t) = x_m$ is defined for any t , and $|\partial_t y_m(t)| = 0 < K$, where K is a positive constant to be determined later. Assume that for some k the point y_k is defined and that $|\partial_t y_k(t)| \leq K$. Then, since $y_{k-1}(t) = (E_{s(t)}(f_{i_{k-1}, t}))^{-1}(y_k(t))$, we have

$$E_{s(t)}(f_{i_{k-1}})(t, y_{k-1}(t)) = y_k(t).$$

We will now consider the different cases in the definition of E_s in (1). Consider the first case, in which we have

$$E_{s(t)}(f_{i_{k-1}})(t, y_{k-1}(t)) = s(t)f_{i_{k-1}}(t, y_{k-1}(t)) = y_k(t).$$

After a differentiation, we get

$$\begin{aligned} y'_k(t) &= s'(t)f_{i_{k-1}, t}(y_{k-1}(t)) + s(t)\partial_t f_{i_{k-1}}(t, y_{k-1}(t)) \\ &\quad + s(t)\partial_x f_{i_{k-1}}(t, y_{k-1}(t))y'_{k-1}(t). \end{aligned}$$

Solving for $y'_{k-1}(t)$ and using that $s'(t) = \alpha$, we get

$$y'_{k-1}(t) = \frac{(y'_k(t) - \alpha f_{i_{k-1}, t}(y_{k-1}(t)) - s(t)\partial_t f_{i_{k-1}}(t, y_{k-1}(t)))}{s(t)\partial_x f_{i_{k-1}}(t, y_{k-1}(t))}.$$

Hence

$$(3) \quad |y'_{k-1}(t)| \leq \lambda^{-1}(K + \alpha + (1 + \alpha\varepsilon)\eta).$$

Similar calculations for the other cases in (1) yields the same estimate. Hence, in all cases we will have (3). Since $\lambda > 1$, it is clear that this implies that

$$|y'_{k-1}(t)| \leq K$$

if K satisfies $K \geq (\lambda - 1)^{-1}(\alpha + (1 + \alpha\varepsilon)\eta)$. Therefore, put $K = (\lambda - 1)^{-1}(\alpha + (1 + \alpha\varepsilon)\eta)$.

We have now proved that for any $t \in L$, the points $y_k(t)$ are defined and $|y'_k(t)| \leq K$, but we still do not know how big L is.

What can prevent the neighbourhood L to be large is that for some k and t , the point $y_k(t)$ is not in the interval $J_{i_{k-1}}(t)$. However, as t varies, $y_k(t)$ moves at a speed not larger than K , and at the same time, as t grows, the interval $J_{i_{k-1}}(t)$ expands and we would like to know that the endpoints of $J_{i_{k-1}}(t)$ moves with a speed greater than K , so that $y_k(t)$ cannot escape out from the interval $J_{i_{k-1}}(t)$ as t grows. To show that this is the case we need to consider the four cases in (1).

Let $c(t) = E_{s(t)}(f_{i_{k-1}})(t, b_{i_{k-1}}(t))$ be one of the endpoints of $J_{i_{k-1}}(t)$. (The other endpoint can be treated in the same way, but we will not do so.) In the case that $f_{i_{k-1}}(t, b_{i_{k-1}}(t)) \in \{-1, 1\}$, the interval $J_{i_{k-1}}(t)$ will be maximal at one of it's end points, and the point $y_k(t)$ can therefore not escape from $J_{i_{k-1}}(t)$ at that endpoint. It is therefore sufficient to only consider the other cases.

In the case that

$$c(t) = E_{s(t)}(f_{i_{k-1}})(t, b_{i_{k-1}}(t)) = s(t)f_{i_{k-1}}(t, b_{i_{k-1}}(t))$$

we have that

$$\begin{aligned} c'(t) &= s'(t)f_{i_{k-1}}(t, b_{i_{k-1}}(t)) + s(t)\partial_t f_{i_{k-1}}(t, b_{i_{k-1}}(t)) \\ &\quad + s(t)\partial_x f_{i_{k-1}}(t, b_{i_{k-1}}(t))b'_{i_{k-1}}(t). \end{aligned}$$

We estimate each of the three terms separately:

$$\begin{aligned} |s'(t)f_{i_{k-1}}(t, b_{i_{k-1}}(t))| &\geq \alpha\delta, \\ |s(t)\partial_t f_{i_{k-1}}(t, b_{i_{k-1}}(t))| &\leq (1 + \alpha\varepsilon)\eta, \\ |s(t)\partial_x f_{i_{k-1}}(t, b_{i_{k-1}}(t))b'_{i_{k-1}}(t)| &\leq (1 + \alpha\varepsilon)\Lambda\zeta. \end{aligned}$$

This yields

$$|c'(t)| \geq \alpha\delta - (1 + \alpha\varepsilon)(\eta + \Lambda\zeta)$$

and we get the same estimate in the cases

$$c(t) = E_{s(t)}(f_{i_{k-1}})(t, b_{i_{k-1}}(t)) = \frac{s(t) + 1}{2}f_{i_{k-1}}(t, b_{i_{k-1}}(t)) \pm \frac{s(t) - 1}{2}.$$

Now, $|c'(t)| > K \geq |y'_k(t)|$ if

$$\begin{aligned} \alpha\delta - (1 + \alpha\varepsilon)(\eta + \Lambda\zeta) &> \frac{1}{\lambda - 1}(\alpha + (1 + \alpha\varepsilon)\eta) \\ &\Leftrightarrow \\ \alpha\left(\delta - \frac{1}{\lambda - 1}\right) &> (1 + \alpha\varepsilon)\left(\frac{\lambda\eta}{\lambda - 1} + \Lambda\zeta\right). \end{aligned}$$

From this, it appears that if

$$\delta > \frac{1}{\lambda - 1},$$

then there exists a number α_0 such that $|c'(t)| > |y'_k(t)|$ for $\alpha \geq \alpha_0$ provided that $\alpha\varepsilon \leq 1$.

Moreover, since we require that $T_t, s([-1, 1]) \subset [-1, 1]$, the mappings E_s will only be defined for $1 \leq s \leq s_0$, where $s_0 > 1$ only depends on f_t . Hence we also require that $1 + \alpha\varepsilon \leq s_0$. In conclusion, the conclusion of the proposition holds provided

$$\varepsilon < \frac{\min\{1, s_0 - 1\}}{\alpha}. \quad \square$$

6. PROOF OF THEOREM 1

The proof is a modification of Schnellmann's proof of his result. We shall therefore make use of several of the lemmata that are found in Schnellmann's paper [2]. For the proof of these lemmata we refer to Schnellmann's paper. However, we shall repeat here the steps in the proof of Schnellmann's result that coincide with our proof of Theorem 1.

6.1. Densities of Invariant Measures. We first note that the Assumption 5 implies that T_a is weakly covering, and this implies that the density ϕ_a of μ_a satisfies

$$\phi_a(x) \geq \gamma_a > 0,$$

for some γ_a . This is Lemma 4.3 in Liverani's paper [5]. In fact, the proof of that lemma gives

$$(4) \quad \gamma_a \geq 2^{-2} \|T_a\|_\infty^{-N}.$$

It is also clear from Rychlik's paper [6] that ϕ_a is of bounded variation for all a , and that there is a uniform bound on the variation of ϕ_a . In particular, the functions ϕ_a are uniformly bounded. Hence there exists a constant C_0 such that

$$\gamma_a \leq \phi_a(x) \leq C_0, \quad \forall x,$$

holds for all $a \in I$.

We also note that if T_a satisfies Assumption 5, then so does $E_s T_a$ for any $s \geq 1$, where $E_s T_a$ is defined in Section 5. This follows immediately since $T(P) \subset (E_s T_a)(P)$ for any $P \in \mathcal{P}(a) = \mathcal{P}(E_s T_a)$. Note however, that we do not necessarily have $T^n(P) \subset (E_s T_a)^n(P)$ for $n > 1$. This is the only reason that we use Assumption 5 instead of just assuming that T is weakly covering.

By Lemma 4.3 in [5], it then follows that $E_s T_a$ has an invariant probability measure $\mu_{a,s}$ that is equivalent to Lebesgue measure.

Let $\phi_{a,s}$ denote the density of $\mu_{a,s}$. Using the explicit bound on γ_a from below (4), we can conclude that

$$(5) \quad \frac{1}{2}\gamma_a \leq \phi_{a,s}(x) \leq C_0, \quad \forall x,$$

holds for all $a \in I$ and $1 \leq s \leq s_0$.

It would be useful if (5) would give a bound from below that is uniform in a , since then Assumption 3 would be satisfied, but this need not be the case. However, we can take a subset I' of I of almost full measure and a $\gamma > 0$ such that

$$(6) \quad \gamma \leq \phi_{a,s}(x) \leq \gamma^{-1}, \quad \forall x,$$

holds for all $a \in I'$ and $1 \leq s \leq s_0$. We then have Assumption 3 for I' replaced by I , and by working with I' instead of I , we can perform the proof in the same way as if (6) had been satisfied for all $a \in I$.

The possibility to work with I' instead of I was remarked already by Schnellmann, see Remark 2 in [2].

Instead of working with I' we shall however work with I and assume that (6) holds for all $a \in I$; This is mostly a typographical difference, and one can throughout the proof exchange I with I' . Subintervals \tilde{I} of I should be replaced with $\tilde{I} \cap I'$, et c.

6.2. Switching from the parameter space to the phase space. Assumption 1 was used by Schnellmann to switch from integrals over the parameter space to integrals over the phase space. We shall also need to do so, but we will need that Assumption 1 holds also for the mappings $E_s T_a$ with $s = 1 + (a - a_0)\alpha$. We can achieve this as follows.

Assumption 2 implies Assumption 1, as previously mentioned. Hence, $|\xi'_j(a)|$ grows axponentially fast with j . If $T_a^j(X(a))$ is typical, then so is $X(a)$. Therefore, working with $\tilde{X}(a) = T_a^j(X(a))$ instead of $X(a)$, we can achive that $|\xi'_1(a)|$ is as large as we desire. Since

$$\frac{\sup_{a,x} |\partial_a(E_s T_a(x))|}{\lambda - 1} + 2L$$

is bounded, Assumption 2 will be satisfied for $E_s T_a$, and hence also Assumption 1.

6.3. Typical points. We let \mathcal{B} denote the set of open sub intervals of $[0, 1]$ with rational endpoints. The strategy of the proof is to show that there is a constant C such that for any $B \in \mathcal{B}$, the function

$$F_n(a) = \frac{1}{n} \sum_{j=1}^n \chi_B(\xi_j(a))$$

satisfies

$$(7) \quad \limsup_{n \rightarrow \infty} F_n(a) \leq C|B|, \quad \text{for a.e. } a \in I.$$

This is sufficient to prove Theorem 1, since then there is a set of parameters of full measure for which $\limsup F_n(a) \leq C|B|$ holds for every B , and this

implies that any weak accumulation point of

$$\frac{1}{n} \sum_{j=1}^n \delta_{\xi_j(a)}$$

is an absolutely continuous invariant measure with density bounded by C .

At this point, we shall assume that the constant m appearing in Assumption 6 is equal to 1. If, instead of T_a , we consider the family T_a^m , then, as we shall see below, we can prove that $X(a)$ is typical with respect to (T_a^m, μ_a) for almost all a by proving (7) for T_a^m instead of T_a . This result then holds for all of the points $X(a), T_a(X(a)), \dots, T_a^{m-1}(X(a))$, so that (7) holds for T_a as well. Hence, we may assume that $m = 1$.

To prove (7), Schnellmann used a lemma by Björklund and Schnellmann [1]: It is sufficient to show that for all large integers h there is a constant C_1 and an integer $n_{h,B}$, growing at most exponentially fast with h , such that

$$\int_I F_n(a)^h da \leq C_1(C|B|)^h,$$

for all $n \geq n_{h,B}$.

We can write

$$(8) \quad \int_I F_n(a)^h da = \sum_{1 \leq j_1, \dots, j_h \leq n} \frac{1}{n^h} \int_I \chi_B(\xi_{j_1}(a)) \cdots \chi_B(\xi_{j_h}(a)) da.$$

The idea is then to compare the integral over the parameters with integrals over the phase space $[0, 1]$ with respect to μ_a , and use mixing to achieve the desired estimate. Indeed, for a fixed a , there is a set A and a number k such that $T_a^k: A \rightarrow A$ is mixing of any order (see [7]). We therefore have

$$(9) \quad \int_A \chi_B(T_a^{kj_1}(x)) \cdots \chi_B(T_a^{kj_h}(x)) d\mu_a(x) \rightarrow \mu_a(B)^h \mu_a(A) \leq (\gamma^{-1}|B|)^h,$$

as $j_i \rightarrow \infty$ and $\min_{i \neq l} |j_i - j_l| \rightarrow \infty$.

By comparing the integrals in (8) and (9), we shall prove the following.

Proposition 2. *The set I of parameters can be covered by countably many intervals $\tilde{I} \subset I$, such that for each \tilde{I} there is a constant C and numbers $n_{h,B}$, growing at most exponentially fast with h , such that*

$$\int_{\tilde{I}} \chi_B(\xi_{j_1}(a)) \cdots \chi_B(\xi_{j_h}(a)) da \leq (C|B|)^h,$$

for all (j_1, \dots, j_h) with $\sqrt{n} \leq j_1 < j_2 < \cdots < j_h < n - \sqrt{n}$ and $j_i - j_{i-1} \geq \sqrt{n}$, where $n \geq n_{h,B}$.

Schnellmann proved the above proposition using also Assumption 4.

Let us now see how Proposition 2 finishes the proof of Theorem 1. The number of h -tuples (j_1, \dots, j_h) not satisfying the assumptions of Proposition 2, that is, the number of increasing h -tuples satisfying $j_1 < \sqrt{n}$, $j_h > n - \sqrt{n}$ or $j_{k+1} - j_k < \sqrt{n}$ for some k , is at most $2hn^{h-1/2}$. Hence, by (8) and Proposition 2 we get

$$\int_{\tilde{I}} F_n(a)^h da \leq (C|B|)^h + \frac{2h}{\sqrt{n}} |\tilde{I}| \leq 2(C|B|)^h$$

if $n \geq n_{h,B}$ and

$$n \geq \frac{4h^2|\tilde{I}|^2}{(C|B|)^{2h}}.$$

This implies that (7) holds for almost every a in \tilde{I} . Hence, it remains only to prove Proposition 2.

6.4. Proof of Proposition 2. We shall first state the following lemma. Suppose $J = [a_0, a_1] \subset \tilde{I}$. Consider $t \mapsto T_{a_0+t}$ and take α according to Proposition 1. Note that we can choose α to be independent of J .

Put $s(a) = 1 + (a - a_0)\alpha$. In the following lemma, we will consider the mappings $E_{s(a)}T_a$. For a partition \mathcal{Q} , we denote by $\mathcal{Q}|J$ the partition \mathcal{Q} restricted to the interval J , that is the set of non-empty intersections $\omega \cap J$, with $\omega \in \mathcal{Q}$.

Lemma 1. *Assume that Assumption 2 holds. There is an integer q and a constant C_2 such that if $J \subset \tilde{I}$ is of length about $1/n$, then there is a mapping*

$$\mathcal{U}_J: \mathcal{Q}_n|J \rightarrow \mathcal{P}((E_{s(a_1)}T_{a_1})^n)$$

that is at most q -to-one, and we have

$$d(\xi_j(\omega), (E_{s(a_1)}T_{a_1})^j(\mathcal{U}_J(\omega))) \leq C_2/n, \quad 0 \leq j \leq n - \sqrt{n}$$

and

$$|\omega| \leq C_2|\mathcal{U}_J(\omega)|.$$

This lemma is the main difference compared to Schnellmann's paper. Schnellmann proved a corresponding lemma (Lemma 3.1 in his paper) using also Assumption 4.

Proof. In fact, Lemma 1 follows from Lemma 3.1 in [2] combined with Proposition 1 as follows. Let $S_a = E_{s(a)}T_a$ with $s(a) = 1 + (a - a_0)\alpha$. Then Assumption 2 holds for the family S_a , as noted in Section 6.2.

According to (6), Assumption 3 is satisfied for the family S_a .

By Proposition 1, the family S_a satisfies the assumptions of Lemma 3.1 of [2]. The statement of that Lemma is exactly what is to be proved. \square

From now on, we will work a lot with $E_{s(a_1)}T_{a_1}$. Therefore, to simplify the notation, we put $T = E_{s(a_1)}T_{a_1}$.

Take t_0 such that $2^{1/t_0} \leq \sqrt{\lambda}$ and let

$$\delta(a) = \min\{|\omega| : \omega \in \mathcal{P}_{t_0}(a)\}.$$

It is clear that $\delta(a)$ depends continuously on a , so δ is locally bounded away from zero.

We show that there is a constant C and $n_{h,B}$ such that if $J \subset \tilde{I}$ is an interval of length between $1/(2n)$ and $1/n$, then

$$(10) \quad \int_J \chi_B(\xi_{j_1}(a)) \cdots \chi_B(\xi_{j_h}(a)) da \leq |J|(C|B|)^h,$$

for $n \geq n_{h,B}$ and (j_1, \dots, j_h) satisfying the assumptions of Proposition 2. By covering \tilde{I} with such intervals J , this implies the statement of Proposition 2.

Let

$$\Omega_J = \{\omega \in \mathcal{Q}_n|J : \xi_{j_i}(\omega) \cap B \neq \emptyset, \forall 1 \leq i \leq h\},$$

We show that

$$(11) \quad |\cup \Omega_J| \leq |J|(C|B|)^h,$$

which implies (10), since

$$\{a \in J : \chi_B(\xi_{j_1}(a)) \cdots \chi_B(\xi_{j_h}(a)) = 1\} \subset \cup \Omega_J.$$

Let J_X be the interval of length $|X(J)| + 3C_2/n$ and concentric with $X(J)$. Put

$$\Omega = \{\omega \in \mathcal{P}_n(a_1) | J_X : T^{j_i}(\omega) \cap 2B \neq \emptyset, 1 \leq i \leq h\}.$$

Now, using Lemma 1, we have

$$\cup \mathcal{U}_J(\Omega_J) \subset \cup \Omega$$

and

$$|\cup \Omega_J| \leq C_2 |\cup \mathcal{U}_J(\Omega_J)| \leq C_2 |\cup \Omega|.$$

Take τ such that $\Lambda^{-\tau} \leq |B|/2$ and assume that $n \geq \tau^2$. Then $j_i - j_{i-1} \geq \tau$. Let $\Omega_0 = \{J_X\}$ and

$$\Omega_i = \{\omega \in \mathcal{P}_{j_i+\tau}(a_1) | \cup \Omega_{i-1} : T^{j_i}(\omega) \cap 2B \neq \emptyset\}, \quad 1 \leq i \leq h.$$

We then have $\cup \Omega \subset \cup \Omega_h$.

By the way τ was chosen, we have $|T^{j_i}(\omega)| \leq |B|/2$ for all $\omega \in \mathcal{P}_{j_i+\tau}(a_1)$. Therefore

$$\cup \Omega_i \subset \{x \in \Omega_{i-1} : T^{j_i}(x) \in 3B\}.$$

We let ϕ denote the density of the absolutely continuous invariant measure of T . By the invariance of the density ϕ , we have

$$\phi(x) = \sum_{T^k(y)=x} \frac{\phi(y)}{|(T^k)'(y)|}.$$

Hence, using (6), which states that $\gamma \leq \phi \leq \gamma^{-1}$, we get

$$\sum_{T^k(y)=x} \frac{1}{|(T^k)'(y)|} \leq \gamma^{-2} \quad \text{for a.e. } x.$$

It therefore follows that

$$\begin{aligned} |T^j(\{x \in \omega : T^{j+k}(x) \in 3B\})| &= \int_{3B} \sum_{\substack{x \in \omega, \\ T^{j+k}(x)=y}} \left| \frac{(T^j)'(x)}{(T^{j+k})'(x)} \right| dy \\ &\leq \int_{3B} \sum_{T^k(x)=y} \frac{1}{|(T^k)'(x)|} dy \leq 3\gamma^{-2}|B|. \end{aligned}$$

There is a constant C_3 such that

$$\left| \frac{(T^j)'(x_1)}{(T^j)'(x_2)} \right| \leq C_3$$

holds whenever $x_1, x_2 \in \omega \in \mathcal{P}_j(a_1)$. This is just a standard distortion estimate. See Lemma 4.1 in Schnellmann's paper for a more general result.

If $|T^j(\omega)| \geq \delta_0$, then we get

$$(12) \quad \begin{aligned} |\{x \in \omega : T^{j+k}(x) \in 3B\}| &\leq C_3 \frac{|T^j(\{x \in \omega : T^{j+k}(x) \in 3B\})|}{|T^j(\omega)|} |\omega| \\ &\leq \frac{3\gamma^{-2}C_3}{\delta} |B| |\omega| = C_4 |B| |\omega|. \end{aligned}$$

This gives us enough control over those ω that satisfy $|T^j(\omega)| \geq \delta_0$. We will also need some control of those ω that do not satisfy this requirement, and we introduce the following set of exceptional cylinders

$$F_i = \{\omega \in \mathcal{P}_{j_{i+1}}(a_1) \mid \cup \Omega_i : \exists \tilde{\omega} \in \mathcal{P}_{j_i+k}(a_1) \mid \cup \Omega_i, \tau \leq k \leq j_{i+1} - j_i, \\ \text{s.t. } \tilde{\omega} \supset \omega \text{ and } |T^{j_i+k}(\tilde{\omega})| \geq \delta_0\}.$$

The following lemma from Schnellmann's paper gives us enough control of the sets F_i . The proof is in [2].

Lemma 2. *There are numbers $n_{h,B}$, growing at most exponentially in h , such that*

$$|F_i| \leq \frac{(C_4|B|)^h \mid \cup \Omega_0 \mid}{h},$$

for all $0 \leq i \leq h-1$ and $n \geq n_{h,B}$.

If $\omega \in \Omega_i \setminus F_i$, then for some k we have $|T^{j_i+k}(\omega)| \geq \delta_0$ and we get by (12) that

$$|\{x \in \omega : T^{j_{i+1}}(x) \in 3B\}| \leq C_4 |B| |\omega|.$$

This implies with Lemma 2 that

$$|\cup \Omega_{i+1}| \leq C_4 |B| |\cup (\Omega_i \setminus F_i)| + |F_i| \leq C_4 |B| |\omega_i| + \frac{(C_4|B|)^h \mid \cup \Omega_0 \mid}{h},$$

provided $n \geq n_{h,B}$. Hence

$$|\cup \Omega| \leq |\cup \Omega_h| \leq (C_4|B|)^h \mid \cup \Omega_0 \mid + |h \frac{(C_4|B|)^h \mid \cup \Omega_0 \mid}{h}| \leq 2(C_4|B|)^h \mid \cup \Omega_0 \mid.$$

Since

$$|J_X| = |X(J)| + \frac{3C_2}{n} \leq (6C_2 + \sup_{a \in I} |X'(a)|) |J|$$

and $\Omega_0 = \{J_X\}$, we have

$$|\cup \Omega_0| \leq (6C_2 + \sup_{a \in I} |X'(a)|) |J|.$$

By Lemma 1 we have $|\cup \Omega_J| \leq qC_2 |\cup \Omega|$. Therefore

$$|\cup \Omega_J| \leq (C|B|)^h |J|$$

which implies (11) and finishes the proof of Theorem 1.

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